

Math 4200

Friday October 30

3.2-3.3 isolated zeroes theorem, uniqueness of analytic extensions; begin 3.3 Laurent series. We'll start today by quickly explaining the theorem about multiplying Taylor series in Wednesday's notes, that you used successfully on the quiz.

Announcements:

Consequences of power series for analytic functions:

Theorem (Isolated zeroes theorem). Let

$A \subseteq \mathbb{C}$ be an open, connected set,

$f: A \rightarrow \mathbb{C}$ analytic,

$D(z_0; r) \subseteq A$,

$f(z_0) = 0$.

Then either $f(z) \equiv 0$ in $D(z_0; r)$ or $\exists \delta > 0$ such that $f(z) \neq 0$

$\forall 0 < |z - z_0| < \delta$.

proof: f has convergent Taylor series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad |z - z_0| < r.$$

If all the $a_n = 0$ then $f \equiv 0$ in $D(z_0; r)$. Otherwise let a_N be the first non-zero coefficient in the power series, so

$$f(z) = \sum_{n=N}^{\infty} a_n (z - z_0)^n$$

and factor out the lowest power of $(z - z_0)$ that appears in this series:

$$f(z) = (z - z_0)^N \left(a_N + \sum_{n=N+1}^{\infty} a_n (z - z_0)^{n-N} \right)$$
$$f(z) = (z - z_0)^N g(z)$$

where $g(z_0) = a_N \neq 0$ and $g(z)$ is analytic and hence continuous near z_0 . Thus there exists $\delta > 0$ such that $|z - z_0| < \delta \Rightarrow g(z) \neq 0$. This proves the claim.

QED.

There is a surprising consequence of the isolated zeroes theorem:

Corollary (Unique extensions theorem) Let $A \subseteq \mathbb{C}$ be open and connected; $f, g : A \rightarrow \mathbb{C}$ analytic. Supposed there exists

$$\begin{aligned} & A \subseteq \mathbb{C} \text{ be open and connected,} \\ & f, g : A \rightarrow \mathbb{C} \text{ analytic;} \\ & \{z_k\} \subseteq A, \{z_k\} \rightarrow z_0 \in A, z_k \neq z_0, k \in \mathbb{N}. \\ & f(z_k) = g(z_k) \quad \forall k \end{aligned}$$

Then $f(z) = g(z) \quad \forall z \in A$.

proof: $f - g : A \rightarrow \mathbb{C}$ is analytic and $(f - g)(z_k) = 0 \quad \forall k$. Thus z_0 is a zero of $f - g$ which is not isolated. Thus by the isolated zeroes theorem,

$$(f - g)(z) \equiv 0, \quad \forall z \in D(z_0; r) \subseteq A.$$

(This is already surprising.) Now, consider

$$B := \{z \in A \mid (f - g)^{(n)}(z) = 0 \quad \forall n = 0, 1, 2, \dots\}.$$

We have $D(z_0; r) \subseteq B$ since $(f - g)(z) \equiv 0$ in $D(z_0; r)$.

• B is closed in A because if $\{w_k\} \subseteq B$, $\{w_k\} \rightarrow w \in A$ then $0 = (f - g)^{(n)}(w_k) \rightarrow (f - g)^{(n)}(w) \quad \forall n$.

• B is open in A because if $z_1 \in B$ the Taylor series for f at z_1 is the zero function, so for any $D(z_1; r) \subseteq A$ we also have $D(z_1; r) \subseteq B$.

Thus, since A is connected, $B = A$.

Example (also see one of your homework problems). It is not clear without a lot of work why the Riemann Zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

can be extended as an analytic function (with different formulas), beyond the plane $\operatorname{Re}(s) > 1$ on which the series converges. But in fact, it can be extended as an analytic function on $\mathbb{C} \setminus \{1\}$. The Unique extensions theorem says there's only one possible extension.

3.3 Laurent series. If $f(z)$ is an analytic function on a punctured disk or on an annulus centered at z_0 , then f can be expressed as a power series expansion using non-negative *and* negative powers of $(z - z_0)$. These series are called *Laurent series*

Laurent Series Theorem For $0 \leq R_1 < R_2$ let

$$A = \{z \in \mathbb{C} \mid R_1 < |z - z_0| < R_2\}$$

be an open annulus (or punctured disk in case $R_1 = 0$). Then (1) and (2) below are equivalent, and the uniqueness of Laurent coefficients (3) also holds:

- (1) $f: A \rightarrow \mathbb{C}$ is analytic.
 (2) $f(z)$ has a power series expansion using non-negative and negative powers of $(z - z_0)$:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{m=1}^{\infty} \frac{a_{-m}}{(z - z_0)^m}.$$

$$:= S_1(z) + S_2(z).$$

Here $S_1(z)$ converges for $|z - z_0| < R_2$ and uniformly absolutely for $|z - z_0| \leq r_2 < R_2$. And $S_2(z)$ converges for $|z - z_0| > R_1$, and uniformly for $|z - z_0| \geq r_1 > R_1$.

- (3) The Laurent coefficients a_k , $k \in \mathbb{Z}$ are uniquely determined by f . Specifically, if γ is any p.w. C^1 contour in A , with $I(\gamma, z_0) = 1$, e.g. any circle of radius r , with $R_1 < r < R_2$, then

$$a_k = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} d\zeta$$

In particular the contour integral of f itself has value

$$\int_{\gamma} f(\zeta) d\zeta = 2\pi i a_{-1}.$$

For this reason, the coefficient a_{-1} of $\frac{1}{z - z_0}$ in the Laurent series, is called the *residue* of f at z_0 . (Because it's the only part of the Laurent series you need to know in order to compute the contour integral of f .)

Note: (2) \Rightarrow (1) of the theorem is immediate, since uniform limits of analytic functions are analytic. We'll prove (2) \Rightarrow (3) in today's notes, and then do (1) \Rightarrow (2) on Monday. It will rely on geometric series magic, as did our theorem about Taylor series for analytic functions in disks.

Examples:

1) Consider

$$f(z) = \frac{1}{(z-1)(z+2)} = \frac{1}{3} \left(\frac{1}{z-1} - \frac{1}{z+2} \right).$$

Find the following series expansions for $z_0 = 0$:

- a) Taylor series for $|z| < 1$
- b) Laurent series for $1 < |z| < 2$.
- c) Laurent series for $|z| > 2$.

Use residues from the Laurent series to compute $\int_{\gamma} f(z) dz$ for the three index-one

circles centered at the origin, of radii $\frac{1}{2}$, $\frac{3}{2}$, 3 . Notice that this is reproducing results you already know how to find using the Cauchy integral formula and other means.

- 2a) What is the Laurent series for $z e^{\frac{1}{z}}$ in $\mathbb{C} \setminus \{0\}$?
- 2b) What is the value of

$$\int_{\gamma} z e^{\frac{1}{z}} dz$$

if γ is a closed contour in $\mathbb{C} \setminus \{0\}$, with $I(\gamma; 0) = 1$?

proof of (2) \Rightarrow (3) in the Laurent series theorem:

(2) $f(z)$ has a power series expansion using non-negative and negative powers of $(z - z_0)$:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{m=1}^{\infty} a_{-m} (z - z_0)^{-m} \\ := S_1(z) + S_2(z).$$

Here $S_1(z)$ converges for $|z - z_0| < R_2$ and uniformly absolutely for $|z - z_0| \leq r_2 < R_2$. And $S_2(z)$ converges for $|z - z_0| > R_1$, and uniformly for $|z - z_0| \geq r_1 > R_1$.

(3) The Laurent coefficients a_k , $k \in \mathbb{Z}$ are uniquely determined by f . Specifically, if γ is any p.w. C^1 contour in A , with $I(\gamma, z_0) = 1$, e.g. any circle of radius r , with $R_1 < r < R_2$, then

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In particular the contour integral of f itself has value

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proof: We'll write $f(\zeta) = S_1(\zeta) + S_2(\zeta)$ and just compute the contour integrals above. We'll use the uniform convergence of the series $S_1(\zeta)$, $S_2(\zeta)$ on γ to interchange the integrals with the summations:

Math 4200-001
Week 10-11 concepts and homework
3.2-3.3
Due Friday November 6 at start of class.

3.3 1ab, 4, 6, 8, 13, 15, 17, 18, 19, 20

w10.1a) Use the definitions of even and odd functions to show that if f is analytic in a neighborhood of the origin and if f is even, then its Taylor series at $z_0 = 0$ only contains even powers of z ; and if f is odd the Taylor series only contains odd powers.

w10.1b) Are the same facts true for the Laurent series based at $z_0 = 0$, for even and odd analytic functions defined in annuli concentric to the origin?

w10.2) Let f be an entire function. Suppose $f\left(\frac{1}{n}\right) = \frac{1}{n^2}$ for all positive integers n .

Is it possible for $f(-1)$ to equal -1 ? Explain.

w10.3) Use power series or L'Hopital's rule to find

$$\lim_{z \rightarrow 0} \frac{\cos(z) - 1}{z \sin(z)}$$

w10.4) Continuing the text problem 3.3.4, find the Laurent series for

$$\frac{1}{z(z-1)(z-2)}$$

valid for $|z| > 2$.

w10.4) Which of these functions has a removable singularity at $z = 0$?

a) $\frac{\cos(z) - 1}{z \sin(z)}$ (see w10.2)

b) $\frac{\cos(z) - 1}{z^3}$.